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On the computation of the Lichnerowicz–Jacobi cohomology

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Abstract

Lichnerowicz–Jacobi cohomology of Jacobi manifolds is reviewed. The use of the associated Lie algebroid allows to prove that the Lichnerowicz–Jacobi cohomology is invariant under conformal changes of the Jacobi structure. We also compute the Lichnerowicz–Jacobi cohomology for a large variety of examples.

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1. Introduction

Since their introduction by Lichnerowicz in [18,19], Poisson and Jacobi manifolds have deserved a lot of interest in the mathematical physics literature. Indeed, the need to use more general phase spaces for Hamiltonian systems lead to the consideration of Poisson

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brackets of non-constant rank, and, more than this, brackets which do not satisfy Leibniz rule (Jacobi brackets).

From the viewpoint of differential geometry, both structures are of great interest. The local and global structures of Poisson and Jacobi manifolds were elucidated by several authors ([4,9,11,28]; see also [1,16,25]). A Poisson manifold is basically made of symplectic pieces, but the structure of a Jacobi manifold is more complicated, and it is made of pieces which are contact or locally conformal symplectic manifolds.

The Poisson structure of a Poisson manifold M allows to define some cohomology operators. Indeed, the Poisson bivector of M determines the so-called Lichnerowicz–Poisson cohomology (LP cohomology) and the 1-differentiable Chevalley–Eilenberg cohomology, which can be alternatively described as the cohomologies of two subcomplexes of the Chevalley–Eilenberg complex associated with the Lie algebra of differentiable functions endowed with its Poisson bracket (see [18]). Computation of Poisson cohomology is generally quite difficult. For regular Poisson manifolds and for the Lie–Poisson structure on the dual space of the Lie algebra of a compact Lie group, some results were obtained in [6,7,24,30]. On the other hand, we remark that the kth LP cohomology group has interesting interpretations for the first few values of k. Moreover, these cohomology groups allow to describe important results about the geometric quantization and the deformation quantization of Poisson manifolds (for more information, we refer to [25] and to the recent survey [29]; see also the references therein).

The situation for a Jacobi manifold M is more involved. Note that the Jacobi bracket of functions on M is a linear skew-symmetric 2-differential operator of order 1 or, in other words, a 1-differentiable 2-cochain in the Chevalley–Eilenberg complex of the Lie algebra of functions. Imitating the Poisson case, for a Jacobi manifold, one can consider the representation of the Lie algebra of functions on itself given by the Jacobi bracket. The resultant cohomology, the Chevalley–Eilenberg cohomology, was studied by Guédira and Lichnerowicz [9] and Lichnerowicz [19]. Particularly, they studied the 1-differentiable Chevalley–Eilenberg cohomology, that is, the cohomology of the subcomplex of the Chevalley–Eilenberg complex which consists of the 1-differentiable cochains. But there is a second possibility considering the representation of the Lie algebra of functions on itself given by the action of the Hamiltonian vector fields. The resultant cohomology was termed by the authors, in [14,15], the H–Chevalley–Eilenberg cohomology. As in the case of the Chevalley–Eilenberg complex, one can consider also the cohomology of the subcomplex of the 1-differentiable cochains which was called the Lichnerowicz–Jacobi cohomology, LJ cohomology, for brevity (see [14,15]). For a Poisson manifold, the Chevalley–Eilenberg cohomology and the H–Chevalley–Eilenberg cohomology coincide and the 1-differentiable Chevalley–Eilenberg cohomology is just the LJ cohomology. The H–Chevalley–Eilenberg cohomology and the LJ cohomology of a Jacobi manifold M play an important role in the geometric quantization of M and in the study of the existence of prequantization representations for complex line bundles over M (for more details, see [14,15]).

The LJ cohomology can be also described using the Lie algebroid associated with the Jacobi manifold. Indeed, it is just the Lie algebroid cohomology with trivial coefficients (see [14,15,26]).

In this paper we review this cohomology theory obtaining new properties about it. So, thinking about the intrinsic conformal character of the Jacobi structures (the Hamiltonian vector fields of a Jacobi manifold are conformal Jacobi transformations), we prove that the LJ cohomology is invariant under conformal changes of the Jacobi structure.

Moreover, we compute the LJ cohomology for some relevant examples of Jacobi manifolds: Poisson manifolds, contact manifolds, locally conformal symplectic manifolds and the Jacobi structure of the unit sphere of a finite-dimensional real Lie algebra.

All the manifolds considered in this paper are assumed to be connected. Furthermore, if M is a differentiable manifold, we will denote by $C^\infty(M,\mathbb{R})$ the algebra of C^∞ real-valued functions on M, by $\mathfrak{X}(M)$ the Lie algebra of the vector fields, by $\Omega^k(M)$ the space of k-forms and by $\mathcal{V}^k(M)$ the space of k-vectors.

2. Jacobi manifolds and Lie algebroids

A *Jacobi structure* on a manifold M is a pair (Λ, E), where Λ is a 2-vector and E is a vector field on M satisfying the following properties:

$$[\Lambda,\Lambda] = 2E\wedge\Lambda, \qquad \mathcal{L}_E\Lambda = [E,\Lambda] = 0. \tag{2.1}$$

Here $[\cdot,\cdot]$ denotes the Schouten–Nijenhuis bracket [1,25] and $\mathcal{L}$ is the Lie derivative operator. The manifold M endowed with a Jacobi structure is called a *Jacobi manifold*. A bracket of functions (the *Jacobi bracket*) is defined by

$$\{f,g\} = \Lambda(\mathrm{d}f,\mathrm{d}g) + fE(g) - gE(f) \quad \text{for all } \ f,g\in C^\infty(M,\mathbb{R}).$$

Thus, the space $C^\infty(M,\mathbb{R})$ endowed with this bracket is *a local Lie algebra* in the sense of Kirillov (see [11]). Conversely, a structure of local Lie algebra on $C^\infty(M,\mathbb{R})$ defines a Jacobi structure on M (see [9,11]). If the vector field E identically vanishes then (M,Λ) is a *Poisson manifold*. Jacobi and Poisson manifolds were introduced by Lichnerowicz [18,19].

Examples of Poisson structures are symplectic and Lie–Poisson structures (see [18,28]). Other interesting examples of Jacobi manifolds, which are not in general Poisson manifolds, are the following ones.

Contact manifolds: Let M be a $(2m+1)$-dimensional manifold and η a 1-form on M. We say that η is a contact 1-form if $\eta\wedge(\mathrm{d}\eta)^m \neq 0$ at every point. In such a case (M,η) is termed a *contact manifold* (see e.g. [2,16,19]). If (M,η) is a contact manifold, we define the associated Jacobi structure on M as follows:

$$\Lambda(\alpha,\beta) = \mathrm{d}\eta(\flat^{-1}(\alpha),\flat^{-1}(\beta)), \qquad E = \flat^{-1}(\eta)$$

for all $\alpha,\beta \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^\infty(M,\mathbb{R})$ modules given by $\flat(X) = i_X\,\mathrm{d}\eta + \eta(X)\eta$.

Locally conformal symplectic manifolds: A *locally conformal symplectic* (l.c.s.) *manifold* is a pair (M,Ω), where M is an even-dimensional manifold and Ω is a non-degenerate 2-form such that, for each point $x\in M$, there is an open neighborhood U and a function $f : U\to\mathbb{R}$ satisfying $\mathrm{d}(\mathrm{e}^f\Omega) = 0$, i.e. $(U, \mathrm{e}^f\Omega)$ is a symplectic manifold. If $U = M$ then M is said to be *a globally conformal symplectic* (g.c.s.) *manifold*. Equivalently, (M,Ω) is a l.(g.)c.s. manifold if Ω is a non-degenerate 2-form and there exists a closed (exact) 1-form ω such that $\mathrm{d}\Omega = \omega\wedge\Omega$. The 1-form ω is called the *Lee* 1-*form* of M. It is obvious that the l.c.s. manifolds with Lee 1-form identically zero are just the symplectic manifolds (see e.g. [9,23]).

In a similar way that for contact manifolds, the Jacobi structure (Λ, E) associated to a l.c.s. manifold (M, Ω) with Lee 1-form ω is given by

$$\Lambda(\alpha, \beta) = \Omega(\flat^{-1}(\alpha), \flat^{-1}(\beta)), \qquad E = \flat^{-1}(\omega) \tag{2.2}$$

for all $\alpha, \beta \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X \Omega$ (see [9]).

Unit sphere of a real Lie algebra: Let $(\mathfrak{g}, [\cdot, \cdot])$ be a real Lie algebra of dimension n and let $\bar{\Lambda}$ be the Lie–Poisson 2-vector on the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$. Suppose that $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathfrak{g}$ and that g is the corresponding Riemannian metric on $\mathfrak{g}$. Using the linear isomorphism $\flat_{\langle \cdot, \cdot \rangle} : \mathfrak{g} \to \mathfrak{g}^*$ given by $\flat_{\langle \cdot, \cdot \rangle}(\xi)(\eta) = \langle \xi, \eta \rangle$, for all $\xi, \eta \in \mathfrak{g}$, and the Lie–Poisson structure $\bar{\Lambda}$, we can define a Poisson structure on $\mathfrak{g}$ which we also denote by $\bar{\Lambda}$. Now, we consider the 2-vector Λ' and the vector field E' on $\mathfrak{g}$ given by

$$\Lambda' = \bar{\Lambda} - A \wedge i_\alpha \bar{\Lambda}, \qquad E' = i_\alpha \bar{\Lambda}, \tag{2.3}$$

where A is the radial vector field on $\mathfrak{g}$ and α is the 1-form defined by $\alpha(X) = g(X, A)$, for $X \in \mathfrak{X}(\mathfrak{g})$. Thus, the pair (Λ', E') induces a Jacobi structure on $\mathfrak{g}$. Moreover, if $S^{n-1}(\mathfrak{g})$ is the unit sphere in $\mathfrak{g}$, it follows that the restrictions Λ and E to $S^{n-1}(\mathfrak{g})$ of Λ' and E', respectively, are tangent to $S^{n-1}(\mathfrak{g})$. Therefore, the pair (Λ, E) defines a Jacobi structure on $S^{n-1}(\mathfrak{g})$ (see [20]). In fact, (Λ, E) is a Poisson structure if and only if $\langle \cdot, \cdot \rangle$ is invariant under the adjoint representation $\mathrm{Ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

On the other hand, let (M, Λ, E) be a Jacobi manifold. Define a homomorphism of $C^\infty(M, \mathbb{R})$ modules $\#_\Lambda : \Omega^1(M) \to \mathfrak{X}(M)$ by

$$(\#_\Lambda(\alpha))(\beta) = \Lambda(\alpha, \beta) \tag{2.4}$$

for $\alpha, \beta \in \Omega^1(M)$. This homomorphism can be extended to a homomorphism, which we also denote by $\#_\Lambda$, from the space $\Omega^k(M)$ onto the space $\mathcal{V}^k(M)$ by putting

$$\#_\Lambda(f) = f, \quad \#_\Lambda(\alpha)(\alpha_1, \ldots, \alpha_k) = (-1)^k \alpha(\#_\Lambda(\alpha_1), \ldots, \#_\Lambda(\alpha_k)) \tag{2.5}$$

for $f \in C^\infty(M, \mathbb{R})$, $\alpha \in \Omega^k(M)$ and $\alpha_1, \ldots, \alpha_k \in \Omega^1(M)$.

If f is a C^∞ real-valued function on a Jacobi manifold M, the vector field X_f defined by

$$X_f = \#_\Lambda(\mathrm{d}f) + fE$$

is called the *Hamiltonian vector field* associated with f. Now, for every $x \in M$, we consider the subspace $\mathcal{F}_x$ of T_xM generated by all the Hamiltonian vector fields evaluated at the point x. In other words, $\mathcal{F}_x = (\#_\Lambda)_x(T_x^*M) + \langle E_x \rangle$. Since $\mathcal{F}$ is involutive, one easily follows that $\mathcal{F}$ defines a generalized foliation in the sense of Sussmann [22], which is called the *characteristic foliation* (see [4,9]). Moreover, the Jacobi structure of M induces a Jacobi structure on each leaf which is a contact or a l.c.s. structure ([4,9]). If M is a Poisson manifold then the characteristic foliation of M is just the *canonical symplectic foliation* of M (see [25,28]).

To finish this section, we recall the definition of the Lie algebroid structure associated with a Jacobi manifold.

Let (M, Λ, E) be a Jacobi manifold. In [10], the authors obtain a Lie algebroid structure $([\![\cdot,\cdot]\!]_{(\Lambda,E)}, (\#_\Lambda, E))$ on the vector bundle $J^1(M,\mathbb{R}) \cong T^*M \times \mathbb{R} \to M$, where the homomorphism of $C^\infty(M,\mathbb{R})$ modules $(\#_\Lambda, E) : \Gamma(J^1(M,\mathbb{R})) \cong \Omega^1(M) \times C^\infty(M,\mathbb{R}) \to \mathfrak{X}(M)$ defined by

$$(\#_\Lambda, E)(\alpha, f) = \#_\Lambda(\alpha) + fE \tag{2.6}$$

is the anchor map and the Lie bracket $[\![\cdot,\cdot]\!]_{(\Lambda,E)} : (\Omega^1(M) \times C^\infty(M,\mathbb{R}))^2 \to \Omega^1(M) \times C^\infty(M,\mathbb{R})$ is given by

$$\begin{aligned}[\![(\alpha, f), (\beta, g)]\!]_{(\Lambda,E)} = (&\mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha - \mathrm{d}(\Lambda(\alpha,\beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha \\ &- i_E(\alpha\wedge\beta), \alpha(\#_\Lambda(\beta)) + \#_\Lambda(\alpha)(g) - \#_\Lambda(\beta)(f) \\ &+ fE(g) - gE(f)).\end{aligned} \tag{2.7}$$

In fact, if Λ is a 2-vector and E is a vector field on a manifold M, we can consider the homomorphism of $C^\infty(M,\mathbb{R})$ modules $(\#_\Lambda, E) : \Omega^1(M) \times C^\infty(M,\mathbb{R}) \to \mathfrak{X}(M)$ and the bracket $[\![\cdot,\cdot]\!]_{(\Lambda,E)} : (\Omega^1(M) \times C^\infty(M,\mathbb{R}))^2 \to \Omega^1(M) \times C^\infty(M,\mathbb{R})$ defined as in (2.6) and (2.7), respectively. Then, (Λ, E) is a Jacobi structure on M if and only if $([\![\cdot,\cdot]\!]_{(\Lambda,E)}, (\#_\Lambda, E))$, is a Lie algebroid structure on $J^1(M,\mathbb{R})$.

In the particular case when (M, Λ) is a Poisson manifold we recover, by projection on the first factor, the Lie algebroid associated to M (see [1,3,5,25]).

3. Lichnerowicz–Jacobi cohomology and conformal changes

Let (M, Λ, E) be a Jacobi manifold. Denote by $\{\cdot,\cdot\}$ its associated bracket. We consider the cohomology complex $(C^*_{\mathrm{HCE}}(M), \partial_H)$ of the Lie algebra $(C^\infty(M,\mathbb{R}), \{\cdot,\cdot\})$ relative to the representation defined by the Hamiltonian vector fields, that is,

$$C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R}), \quad (f, g) \to X_f(g).$$

Its corresponding cohomology $H^*_{\mathrm{HCE}}(M)$ is called the *H–Chevalley–Eilenberg cohomology* associated to M (see [13–15]). Note that for a Poisson manifold M, $H^*_{\mathrm{HCE}}(M)$ is the *Chevalley–Eilenberg cohomology* of the Lie algebra $(C^\infty(M,\mathbb{R}), \{\cdot,\cdot\})$ (see [18]). However, for arbitrary Jacobi manifolds, the Chevalley–Eilenberg cohomology (which is defined with respect to the representation given by the Jacobi bracket [19]) does not coincide in general with the H–Chevalley–Eilenberg cohomology.

An interesting subcomplex of the H–Chevalley–Eilenberg complex is the complex of the 1-differentiable cochains. A k-cochain $c^k \in C^k_{\mathrm{HCE}}(M)$ is said to be 1-*differentiable* if it is defined by a k-linear skew-symmetric differential operator of order 1. Then, we can identify the space $\mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ with the space of all 1-differentiable k-cochains $C^k_{\mathrm{HCE\text{-}1diff}}(M)$ using the isomorphism $j^k : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \to C^k_{\mathrm{HCE\text{-}1diff}}(M)$ given by

$$\begin{aligned}j^k(P, Q)(f_1, \ldots, f_k) = P(\mathrm{d}f_1, \ldots, \mathrm{d}f_k) \\ + \sum_{q=1}^{k}(-1)^{q+1} f_q Q(\mathrm{d}f_1, \ldots, \widehat{\mathrm{d}f_q}, \ldots, \mathrm{d}f_k).\end{aligned} \tag{3.1}$$

Under this identification, we have a new cohomology complex $(\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), \sigma)$, where the cohomology operator σ is defined by

$$\sigma(P, Q) = (-[\Lambda, P] + kE \wedge P + \Lambda \wedge Q, [\Lambda, Q] - (k-1)E \wedge Q + [E, P]) \tag{3.2}$$

for all $(P, Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$. The cohomology of this complex will be called the *Lichnerowicz–Jacobi cohomology* (*LJ cohomology*) *of* M and denoted by $H^*_{\mathrm{LJ}}(M, \Lambda, E)$ or simply by $H^*_{\mathrm{LJ}}(M)$ if there is not danger of confusion (see [14,15]). This cohomology is a generalization of the Lichnerowicz–Jacobi cohomology introduced in [12,13]. In fact, the former one is the cohomology of the subcomplex of the pairs $(P, 0)$, where P is invariant by E. For this reason, we retain the name.

Moreover, if $(J^1(M, \mathbb{R}), [\![\cdot, \cdot]\!]_{(\Lambda, E)}, (\#_\Lambda, E))$ is the Lie algebroid over M (see Section 2), then, in [26] it is proved that the LJ cohomology of M is just the Lie algebroid cohomology of $J^1(M, \mathbb{R})$ with trivial coefficients (for the definition of the Lie algebroid cohomology see, for instance, [21]).

Next, we will prove that the LJ cohomology is invariant under conformal changes.

Let (Λ, E) be a Jacobi structure on M. A *conformal change* of (Λ, E) is a new Jacobi structure (Λ_a, E_a) on M defined by

$$\Lambda_a = a\Lambda, \qquad E_a = X_a = \#_\Lambda(\mathrm{d}a) + aE, \tag{3.3}$$

a being a positive C^∞ real-valued function on M (see [4,9]). Moreover, we have the following theorem.

Theorem 3.1. *The LJ cohomology is invariant under conformal changes of the Jacobi structure.*

Proof. Let (M, Λ, E) be a Jacobi manifold and (Λ_a, E_a) a conformal change of the Jacobi structure (Λ, E). We define the isomorphism of vector bundles $\phi : T^*M \times \mathbb{R} \to T^*M \times \mathbb{R}$ by

$$\phi(\alpha_x, \lambda) = \left(\frac{1}{a(x)}\alpha_x + \lambda\, \mathrm{d}\left(\frac{1}{a}\right)(x), \frac{\lambda}{a(x)}\right) \quad \text{for } \alpha_x \in T^*_x M \text{ and } \lambda \in \mathbb{R}. \tag{3.4}$$

A direct computation, using (2.6), (2.7), (3.3) and (3.4), proves that ϕ defines an isomorphism between the Lie algebroids $(T^*M \times \mathbb{R}, [\![\cdot, \cdot]\!]_{(\Lambda, E)}, (\#_\Lambda, E))$ and $(T^*M \times \mathbb{R}, [\![\cdot, \cdot]\!]_{(\Lambda_a, E_a)}, (\#_{\Lambda_a}, E_a))$ associated with the Jacobi structures (Λ, E) and (Λ_a, E_a), respectively. Therefore (see [21]), it follows that $H^k_{\mathrm{LJ}}(M, \Lambda, E) \cong H^k_{\mathrm{LJ}}(M, \Lambda_a, E_a)$, for all k. □

4. Examples

4.1. Poisson manifolds

Now, let (M, Λ) be a Poisson manifold and let σ be the LJ cohomology operator. Denote by $\bar{\sigma}$ the cohomology operator of the subcomplex of the pairs $(P, 0)$. Under the canonical identification $\mathcal{V}^k(M) \oplus \{0\} \cong \mathcal{V}^k(M)$, we have that $\bar{\sigma}(P) = -[\Lambda, P]$. The cohomology of the complex $(\mathcal{V}^*(M), \bar{\sigma})$ is called the *Lichnerowicz–Poisson cohomology* (LP cohomology) of M and denoted by $H^*_{\mathrm{LP}}(M)$ (see [18,25]).

In [18] (see also [17]), Lichnerowicz has exhibited the relation between the LJ cohomology (the 1-differentiable Chevalley–Eilenberg cohomology in his terminology) and the LP cohomology of a Poisson manifold (M, Λ). In fact, he proves that if $\dim H^k_{\mathrm{LP}}(M) < \infty$, for all k, then the LJ cohomology groups have finite dimension and

$$H^k_{\mathrm{LJ}}(M) \cong \frac{H^k_{\mathrm{LP}}(M)}{\mathrm{Im}\, L^{k-2}} \oplus \ker L^{k-1}, \tag{4.1}$$

where $L^k : H^k_{\mathrm{LP}}(M) \to H^{k+2}_{\mathrm{LP}}(M)$ is the homomorphism given by $L^k[P] = [P \wedge \Lambda]$, for all $[P] \in H^k_{\mathrm{LP}}(M)$.

Symplectic structure: If (M, Ω) is a symplectic manifold of dimension $2m$ and finite type then the map $\#_\Lambda : \Omega^k(M) \to \mathcal{V}^k(M)$ induces an isomorphism between $H^k_{\mathrm{LP}}(M)$ and the de Rham cohomology group $H^*_{\mathrm{dR}}(M)$ (see [18,25]). Under this identification, we have that

$$H^k_{\mathrm{LJ}}(M) \cong \frac{H^k_{\mathrm{dR}}(M)}{\mathrm{Im}\, L^{k-2}} \oplus \ker L^{k-1}, \tag{4.2}$$

where now $L^k : H^k_{\mathrm{dR}}(M) \to H^{k+2}_{\mathrm{dR}}(M)$ is the homomorphism given by $L^k([\alpha]) = [\alpha \wedge \Omega]$, for all $[\alpha] \in H^k_{\mathrm{dR}}(M)$ and $0 \le k \le 2m$.

Lie Poisson structure: Let Λ be an *exact Poisson structure* on a manifold M, that is, there exists a vector field A on M such that $\Lambda = \bar{\sigma} A = -\mathcal{L}_A \Lambda$. Then, $H^k_{\mathrm{LJ}}(M) \cong H^k_{\mathrm{LP}}(M) \oplus H^{k-1}_{\mathrm{LP}}(M)$ (see [18]).

Now, suppose that $\mathfrak{g}$ is a real Lie algebra of dimension n and that $\bar{\Lambda}$ is the Lie–Poisson structure on $\mathfrak{g}^*$. Since $\bar{\Lambda}$ is exact, it follows that

$$H^k_{\mathrm{LJ}}(\mathfrak{g}^*) \cong H^k_{\mathrm{LP}}(\mathfrak{g}^*) \oplus H^{k-1}_{\mathrm{LP}}(\mathfrak{g}^*). \tag{4.3}$$

Moreover, if $\mathfrak{g}$ is the Lie algebra of a compact Lie group, in [7] the authors proved that

$$H^k_{\mathrm{LP}}(\mathfrak{g}^*) \cong H^k(\mathfrak{g}) \otimes \mathrm{Inv}, \tag{4.4}$$

where $H^*(\mathfrak{g})$ is the cohomology of $\mathfrak{g}$ relative to the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$ and Inv is the algebra of all *Casimir functions* on $\mathfrak{g}^*$, that is, $\mathrm{Inv} = \{f \in C^\infty(\mathfrak{g}^*, \mathbb{R}) / X_f = 0\}$. Therefore, from (4.3) and (4.4), we conclude that for the Lie algebra $\mathfrak{g}$ of a compact Lie group

$$H^k_{\mathrm{LJ}}(\mathfrak{g}^*) \cong (H^k(\mathfrak{g}) \otimes \mathrm{Inv}) \oplus (H^{k-1}(\mathfrak{g}) \otimes \mathrm{Inv}).$$

4.2. *Contact manifolds*

In order to give an explicit description of the LJ cohomology of a contact manifold, first, we will obtain a general result for Jacobi manifolds which relates the de Rham cohomology and the LJ cohomology.

Let (M, Λ, E) be a Jacobi manifold. Denote by $\#_\Lambda : \Omega^k(M) \to \mathcal{V}^k(M)$ the homomorphism of $C^\infty(M, \mathbb{R})$ modules given by (2.4) and (2.5). Then, we have (see [12,13]):

$$\mathcal{L}_E(\#_\Lambda(\alpha)) = \#_\Lambda(\mathcal{L}_E\alpha),$$
$$-[\Lambda, \#_\Lambda(\alpha)] + kE \wedge \#_\Lambda(\alpha) = -\#_\Lambda(\mathrm{d}\alpha) + \#_\Lambda(i_E\alpha) \wedge \Lambda \tag{4.5}$$

for all $\alpha \in \Omega^k(M)$. Using (2.1), (3.2) and (4.5), we deduce the following result.

Proposition 4.1. *Let (M, Λ, E) be a Jacobi manifold and $\tilde{F}^k : \Omega^k(M) \oplus \Omega^{k-1}(M) \to \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ the homomorphism of $C^\infty(M, \mathbb{R})$ modules defined by*

$$\tilde{F}^k(\alpha, \beta) = (\#_\Lambda(\alpha) + E \wedge \#_\Lambda(\beta), -\#_\Lambda(i_E\alpha) + E \wedge \#_\Lambda(i_E\beta)), \tag{4.6}$$

for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$. Then, the homomorphisms $\tilde{F}^k$ induce a homomorphism of complexes $\tilde{F} : (\Omega^*(M), -\mathrm{d}) \oplus (\Omega^{*-1}(M), \mathrm{d}) \to (\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), \sigma)$. Thus, if $H^*_{\mathrm{dR}}(M)$ is the de Rham cohomology of M, we have the corresponding homomorphism in cohomology $\tilde{F} : H^*_{\mathrm{dR}}(M) \oplus H^{*-1}_{\mathrm{dR}}(M) \to H^*_{\mathrm{LJ}}(M)$.

Now, let (M, η) be a contact manifold and (Λ, E) its associated Jacobi structure. The isomorphism of $C^\infty(M, \mathbb{R})$ modules $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ given by $\flat(X) = i_X(\mathrm{d}\eta) + \eta(X)\eta$, can be extended to a mapping, which we also denote by $\flat$, from the space $\mathcal{V}^k(M)$ onto the space $\Omega^k(M)$ by putting $\flat(X_1 \wedge \cdots \wedge X_k) = \flat(X_1) \wedge \cdots \wedge \flat(X_k)$, for all $X_1, \ldots, X_k \in \mathfrak{X}(M)$. This extension is also an isomorphism of $C^\infty(M, \mathbb{R})$ modules. In fact, it follows that

$$\#_\Lambda\alpha = (-1)^k \flat^{-1}(\alpha) + E \wedge \#_\Lambda(i_E\alpha) \tag{4.7}$$

for $\alpha \in \Omega^k(M)$ (see [13]). Moreover, we have the following result.

Theorem 4.2. *Let (M, η) be a contact manifold of dimension $2m + 1$. Then $H^k_{\mathrm{LJ}}(M) \cong H^k_{\mathrm{dR}}(M) \oplus H^{k-1}_{\mathrm{dR}}(M)$, for all k.*

Proof. Using (4.6) and (4.7) and the fact that $i_E \circ \flat = \flat \circ i_\eta$, we deduce that the homomorphism of $C^\infty(M, \mathbb{R})$ modules $\tilde{G}^k : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \to \Omega^k(M) \oplus \Omega^{k-1}(M)$ given by

$$\tilde{G}^k(P, Q) = ((-1)^k(\flat(P) + \eta \wedge \flat(Q) - \eta \wedge \flat(i_\eta P)), (-1)^{k-1}(\flat(i_\eta P) - \eta \wedge \flat(i_\eta Q)))$$

is just the inverse homomorphism of $\tilde{F}^k : \Omega^k(M) \oplus \Omega^{k-1}(M) \to \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$. □

Remark 4.3. In [17], Lichnerowicz showed that the 1-differentiable Chevalley–Eilenberg cohomology of a contact manifold is trivial (compare this result with Theorem 4.2).

4.3. Locally conformal symplectic manifolds

In this section, we will study the LJ cohomology of a l.c.s. manifold. First, we will obtain some results about a certain cohomology, introduced by Guédira and Lichnerowicz [9], which is associated to an arbitrary differentiable manifold endowed with a closed 1-form.

Let M be a differentiable manifold and ω a closed 1-form on M. Define the cohomology operator d_ω by (see [9])

$$\mathrm{d}_\omega = \mathrm{d} + e(\omega), \tag{4.8}$$

d being the exterior differential and $e(\omega)$ the operator given by

$$e(\omega)(\alpha) = \omega \wedge \alpha \quad \text{for all } \alpha \in \Omega^*(M). \tag{4.9}$$

Denote by $H^*_\omega(M)$ the cohomology of the complex $(\Omega^*(M), \mathrm{d}_\omega)$.

Proposition 4.4. *Let M be a differentiable manifold and ω a closed* 1*-form on M. Then,*

(i) *The differential complex $(\Omega^*(M), \mathrm{d}_\omega)$ is elliptic. Thus, if M is compact the cohomology groups $H^k_\omega(M)$ have finite dimension.*
(ii) *If ω is exact then $H^k_\omega(M) \cong H^k_{\mathrm{dR}}(M)$.*

Proof.

(i) It is easy to check that the differential operators d and d_ω have the same symbol which implies that the complex $(\Omega^*(M), \mathrm{d}_\omega)$ is elliptic.
(ii) A direct computation proves that if $\omega = \mathrm{d}f$, with f a C^∞ real-valued function on M, then the mapping $\phi : H^k_{\mathrm{dR}}(M) \to H^k_\omega(M)$, given by $\phi([\alpha]) = [\mathrm{e}^{-f}\alpha]$, is an isomorphism. □

If the 1-form ω is not exact then, in general, $H^*_\omega(M) \ncong H^*_{\mathrm{dR}}(M)$. In fact, we will show next that if M is compact and ω is non-null and parallel with respect to a Riemannian metric on M, then the cohomology $H^*_\omega(M)$ is trivial. First, we will recall some results proved by Guédira and Lichnerowicz [9] which will be useful in the sequel.

Suppose that M is a compact differentiable manifold of dimension n, that ω is a closed 1-form on M and that g is a Riemannian metric. Consider the vector field U on M characterized by the condition $\omega(X) = g(X, U)$, for all $X \in \mathfrak{X}(M)$. Denote by δ the codifferential operator and by i_U the contraction by the vector field U, that is (see [8]),

$$\delta\alpha = (-1)^{nk+n+1}(\star \circ \mathrm{d} \circ \star)(\alpha),$$
$$i_U(\alpha) = (-1)^{nk+n}(\star \circ e(\omega) \circ \star)(\alpha) \quad \text{for } \alpha \in \Omega^k(M), \tag{4.10}$$

$\star$ being the Hodge star isomorphism. Then, we define the operator $\delta_\omega : \Omega^k(M) \to \Omega^{k-1}(M)$ by (see [9])

$$\delta_\omega = \delta + i_U. \tag{4.11}$$

Now, consider the standard scalar product $\langle\cdot,\cdot\rangle$ on the space $\Omega^k(M)$:

$$\langle\cdot,\cdot\rangle : \Omega^k(M) \times \Omega^k(M) \to \mathbb{R}, \qquad (\alpha, \beta) \mapsto \langle\alpha, \beta\rangle = \int_M \alpha \wedge \star\beta.$$

Then, it is easy to prove that $\langle \mathrm{d}_\omega\alpha, \beta\rangle = \langle\alpha, \delta_\omega\beta\rangle$, for all $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$ (see [9]). Thus, since M is compact and the complex $(\Omega^*(M), \mathrm{d}_\omega)$ is elliptic, we obtain an orthogonal decomposition of the space $\Omega^k(M)$ as follows:

$$\Omega^k(M) = \mathcal{H}^k_\omega(M) \oplus \mathrm{d}_\omega(\Omega^{k-1}(M)) \oplus \delta_\omega(\Omega^{k+1}(M)), \tag{4.12}$$

where $\mathcal{H}^k_\omega(M) = \{\alpha \in \Omega^k(M)/\mathrm{d}_\omega(\alpha) = 0, \delta_\omega(\alpha) = 0\}$ (see [9]). From (4.12), it follows that

$$H^k_\omega(M) \cong \mathcal{H}^k_\omega(M). \tag{4.13}$$

Now, we will prove the announced result about the triviality of the cohomology $H^*_\omega(M)$.

Theorem 4.5. *Let M be a compact differentiable manifold and ω a closed* 1*-form on M, $\omega \neq 0$. Suppose that g is a Riemannian metric on M such that ω is parallel with respect to g. Then, the cohomology $H^*_\omega(M)$ is trivial.*

Proof. Since ω is parallel and non-null it follows that $\|\omega\| = c$, with c constant, $c > 0$. Assume, without the loss of generality, that $c = 1$. Note that if $c \neq 1$, we can consider the Riemannian metric $g' = c^2 g$ and it is clear that the module of ω with respect to g' is 1 and that ω is also parallel with respect to g'.

Under the hypothesis $c = 1$, we have that

$$\omega(U) = 1. \tag{4.14}$$

Using that ω is parallel and that U is Killing, we obtain that (see (4.10) and [8])

$$\mathcal{L}_U = -\delta \circ e(\omega) - e(\omega) \circ \delta, \tag{4.15}$$

$$\delta \circ \mathcal{L}_U = \mathcal{L}_U \circ \delta. \tag{4.16}$$

From (4.8)–(4.11), (4.14) and (4.16), we deduce the following relations:

$$\mathrm{d}_\omega \circ i_U = -i_U \circ \mathrm{d}_\omega + \mathcal{L}_U + \mathrm{Id}, \qquad \delta_\omega \circ i_U = -i_U \circ \delta_\omega, \tag{4.17}$$

$$\mathrm{d}_\omega \circ \mathcal{L}_U = \mathcal{L}_U \circ \mathrm{d}_\omega, \qquad \delta_\omega \circ \mathcal{L}_U = \mathcal{L}_U \circ \delta_\omega, \tag{4.18}$$

where Id denotes the identity transformation.

On the other hand, (4.15) implies that $\langle \mathcal{L}_U \alpha, \alpha\rangle = -\langle \alpha, \mathrm{d} i_U \alpha + i_U\, \mathrm{d}\alpha\rangle = -\langle \alpha, \mathcal{L}_U \alpha\rangle$, for all $\alpha \in \Omega^k(M)$. Thus,

$$\langle \mathcal{L}_U \alpha, \alpha\rangle = 0. \tag{4.19}$$

Now, if $\alpha \in \mathcal{H}^k_\omega(M)$ then, using (4.17), we have that $\mathcal{L}_U \alpha = -\alpha + \mathrm{d}_\omega(i_U \alpha)$. But, by (4.18), we deduce that $\mathcal{L}_U \alpha \in \mathcal{H}^k_\omega(M)$. Therefore (see (4.12)), we obtain that $\mathcal{L}_U \alpha = -\alpha$. Consequently, from (4.19), it follows that $\alpha = 0$. This proves that $\mathcal{H}^k_\omega(M) = \{0\}$ which implies that $H^k_\omega(M) = \{0\}$ (see (4.13)). □

Next, we will obtain some results which relate the LJ cohomology of a l.c.s. manifold M, the cohomology $H^*_\omega(M)$ (ω being the Lee 1-form of M) and the de Rham cohomology of M.

Let (M, Ω) be a l.c.s. manifold with Lee 1-form ω. Suppose that (Λ, E) is the associated Jacobi structure on M and that $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$

modules defined by $\flat(X) = i_X\Omega$. The isomorphism $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ can be extended to a mapping, which we also denote by $\flat$, from the space $\mathcal{V}^k(M)$ onto the space $\Omega^k(M)$ by putting: $\flat(X_1 \wedge \cdots \wedge X_k) = \flat(X_1) \wedge \cdots \wedge \flat(X_k)$, for all $X_1, \ldots, X_k \in \mathfrak{X}(M)$. This extension is also an isomorphism of $C^\infty(M, \mathbb{R})$ modules. In fact, we have that (see [13])

$$\#_\Lambda \alpha = (-1)^k \flat^{-1}(\alpha) \quad \text{for all } \alpha \in \Omega^k(M), \tag{4.20}$$

where $\#_\Lambda : \Omega^k(M) \to \mathcal{V}^k(M)$ is the homomorphism given by (2.4) and (2.5).

Using (2.2), (2.4), (2.5), (4.20) and the fact that $\#_\Lambda(\omega) = -E$, we obtain

$$\#_\Lambda \circ i_E = i_\omega \circ \#_\Lambda, \qquad i_E \circ \flat = -\flat \circ i_\omega. \tag{4.21}$$

Thus, from (4.5), (4.20) and (4.21), we deduce that

$$-\flat[\Lambda, P] + k\omega \wedge \flat(P) = \mathrm{d}\flat(P) - i_E(\flat(P)) \wedge \Omega, \qquad \mathcal{L}_E \flat(P) = \flat(\mathcal{L}_E P) \tag{4.22}$$

for all $P \in \mathcal{V}^k(M)$. Now, suppose that $\bar{F}^k : \Omega^k(M) \to \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ and $\bar{G}^k : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \to \Omega^{k-1}(M)$ are the homomorphisms of $C^\infty(M, \mathbb{R})$ modules defined by

$$\bar{F}^k(\alpha) = (\#_\Lambda \alpha, -\#_\Lambda(i_E \alpha)) \text{ and } \bar{G}^k(P, Q) = (-1)^k(-\flat(Q) + i_E \flat(P))$$

for all $\alpha \in \Omega^k(M)$ and $(P, Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$. Then, using (2.2), (3.2), (4.5), (4.8) and (4.20)–(4.22), we prove that the mappings $\bar{F}^k$ and $\bar{G}^k$ induce an exact sequence of complexes

$$0 \to (\Omega^*(M), \mathrm{d}) \xrightarrow{\bar{F}} (\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), -\sigma) \xrightarrow{\bar{G}} (\Omega^{*-1}(M), -\mathrm{d}_\omega) \to 0,$$

where d is the exterior differential, σ the LJ cohomology operator and d_ω is the operator given by (4.8). Thus, one induces a long exact cohomology sequence

$$\cdots \to H^k_{\mathrm{dR}}(M) \xrightarrow{\bar{F}^k_*} H^k_{\mathrm{LJ}}(M) \xrightarrow{\bar{G}^k_*} H^{k-1}_\omega(M) \xrightarrow{L^{k-1}} H^{k+1}_{\mathrm{dR}}(M) \to \cdots,$$

with connecting homomorphism L^{k-1} defined by $L^{k-1}([\alpha]) = [\alpha \wedge \Omega]$, for all $[\alpha] \in H^{k-1}_\omega(M)$. Therefore, we have the following result.

Theorem 4.6. *Let (M, Ω) be a l.c.s. manifold of finite type with Lee 1-form ω and suppose that the dimension of $H^k_\omega(M)$ is finite, for all k. Then,*

$$H^k_{\mathrm{LJ}}(M) \cong \frac{H^k_{\mathrm{dR}}(M)}{\mathrm{Im}\, L^{k-2}} \oplus \ker L^{k-1}.$$

Using Theorems 4.5 and 4.6 and Proposition 4.4, we deduce the following corollaries.

Corollary 4.7. *Let (M, Ω) be a g.c.s. manifold of finite type with Lee 1-form $\omega = \mathrm{d}f$. Then,*

$$H^k_{\mathrm{LJ}}(M) \cong \frac{H^k_{\mathrm{dR}}(M)}{\mathrm{Im}\, \bar{L}^{k-2}} \oplus \ker \bar{L}^{k-1},$$

for all k, where $H^*_{\mathrm{dR}}(M)$ *is the de Rham cohomology of M and* $\bar{L}^r : H^r_{\mathrm{dR}}(M) \to H^{r+2}_{\mathrm{dR}}(M)$ *is the homomorphism defined by* $\bar{L}^r[\alpha] = [\mathrm{e}^{-f}\alpha \wedge \Omega]$, *for all* $[\alpha] \in H^r_{\mathrm{dR}}(M)$.

Corollary 4.8. *Let* (M, Ω) *be a compact l.c.s. manifold with Lee* 1*-form* ω, $\omega \neq 0$. *Suppose that g is a Riemannian metric on M such that* ω *is parallel with respect to g. Then,*

$$H^k_{\mathrm{LJ}}(M) \cong H^k_{\mathrm{dR}}(M) \quad \text{for all } k.$$

Remark 4.9. If (M, Ω) a g.c.s. manifold with Lee 1-form $\omega = \mathrm{d}f$ and (Λ, E) is the associated Jacobi structure, then the 2-form $\bar{\Omega} = \mathrm{e}^{-f}\Omega$ is symplectic. Thus, the Jacobi structure (Λ, E) is a conformal change of the Poisson structure $\bar{\Lambda}$ on M associated with the symplectic 2-form $\bar{\Omega}$. More precisely, we have that $\Lambda = \mathrm{e}^{-f}\bar{\Lambda}$ and $E = \#_{\bar{\Lambda}}(\mathrm{d}(\mathrm{e}^{-f}))$. Thus, Corollary 4.7 follows directly from (4.2) and Theorem 3.1.

Example 4.10. Let (N, η) be a contact manifold.

1. Consider on the product manifold $M = N \times \mathbb{R}$ the 2-form Ω given by

   $$\Omega = (\mathrm{pr}_1)^*(\mathrm{d}\eta) - (\mathrm{pr}_2)^*(\mathrm{d}t) \wedge (\mathrm{pr}_1)^*(\eta),$$

   where t is the usual coordinate on $\mathbb{R}$ and pr_i $(i = 1, 2)$ are the canonical projections of M onto the first and second factor, respectively. Then, (M, Ω) is a g.c.s. manifold with Lee 1-form $\omega = (\mathrm{pr}_2)^*(\mathrm{d}t)$. Moreover, in this case, the symplectic 2-form $\bar{\Omega} = \mathrm{e}^{-t}\Omega$ is exact which implies that the homomorphism $\bar{L}^r$ is null, for all r. Consequently, using Corollary 4.7, it follows that $H^k_{\mathrm{LJ}}(M) \cong H^k_{\mathrm{dR}}(M) \oplus H^{k-1}_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N) \oplus H^{k-1}_{\mathrm{dR}}(N)$.
2. Assume that N is compact and consider on the product manifold $M = N \times S^1$ the 2-form Ω defined by

   $$\Omega = (\mathrm{pr}_1)^*(\mathrm{d}\eta) - (\mathrm{pr}_2)^*(\theta) \wedge (\mathrm{pr}_1)^*(\eta),$$

   θ being the length element of S^1. Then, (M, Ω) is a l.c.s. manifold with Lee 1-form $\omega = (\mathrm{pr}_2)^*(\theta)$. Furthermore, if h is a Riemannian metric on N, the 1-form ω is parallel with respect to the Riemannian metric g on M given by $g = (\mathrm{pr}_1)^*(h) + \omega \otimes \omega$. Therefore, using Corollary 4.8, we deduce $H^k_{\mathrm{LJ}}(M) \cong H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N) \oplus H^{k-1}_{\mathrm{dR}}(N)$.

4.4. The unit sphere of a real Lie algebra

If $(\mathfrak{g}, [\cdot, \cdot])$ is a real Lie algebra of dimension n endowed with a scalar product $\langle \cdot, \cdot \rangle$, then the unit sphere of $\mathfrak{g}$, $S^{n-1}(\mathfrak{g})$, admits a Jacobi structure (see Section 2). In this section, we will describe the LJ cohomology of the sphere for the case when $\mathfrak{g}$ is the Lie algebra of a compact Lie group.

First, we will prove some results which will be useful in the sequel.

Lemma 4.11. *If* $\xi \in \mathfrak{g}$ *and* $\tilde{\xi} : S^{n-1}(\mathfrak{g}) \times \mathbb{R} \to \mathbb{R}$ *is the real* C^∞*-function given by*

$$\tilde{\xi}(\theta, t) = \mathrm{e}^t \langle \xi, \theta \rangle, \tag{4.23}$$

for all $(\theta, t) \in S^{n-1}(\mathfrak{g}) \times \mathbb{R}$, *then* $\partial\tilde{\xi}/\partial t = \tilde{\xi}$. *Moreover, if* $\{\xi_i\}_{i=1,\dots,n}$ *is a basis of* $\mathfrak{g}$ *we have that the set* $\{\mathrm{d}\tilde{\xi}_i\}_{i=1,\dots,n}$ *is a global basis of the space of* 1*-forms on* $S^{n-1}(\mathfrak{g}) \times \mathbb{R}$.

Proof. Let $F : \mathfrak{g} - \{0\} \to S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ be the diffeomorphism defined by $F(\xi) = (\xi/\|\xi\|, \ln\|\xi\|)$. Then, we deduce that $\tilde{\xi} \circ F = \langle \xi, \cdot \rangle$, for all $\xi \in \mathfrak{g}$, where $\langle \xi, \cdot \rangle : \mathfrak{g} - \{0\} \to \mathbb{R}$ is the real function given by $\langle \xi, \cdot \rangle(\theta) = \langle \xi, \theta \rangle$, for all $\theta \in \mathfrak{g} - \{0\}$. This proves the second assertion of Lemma 4.11.

The equation $\partial\tilde{\xi}/\partial t = \tilde{\xi}$ follows directly from (4.23). □

Now, we will describe the LJ cohomology of $S^{n-1}(\mathfrak{g})$ for the case when $\mathfrak{g}$ is the Lie algebra of a compact Lie group.

Theorem 4.12. *Let* $\mathfrak{g}$ *be the Lie algebra of a compact Lie group G of dimension n. Suppose that* $\langle \cdot, \cdot \rangle$ *is a scalar product on* $\mathfrak{g}$ *and consider on the unit sphere* $S^{n-1}(\mathfrak{g})$ *the induced Jacobi structure. Then* $H^k_{\mathrm{LJ}}(S^{n-1}(\mathfrak{g})) \cong H^k(\mathfrak{g}) \otimes \mathrm{Inv}$, *for all k, where* $H^*(\mathfrak{g})$ *is the cohomology of* $\mathfrak{g}$ *relative to the trivial representation of* $\mathfrak{g}$ *on* $\mathbb{R}$ *and* Inv *is the Lie subalgebra of* $C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})$ *defined by* $\mathrm{Inv} = \{\varphi \in C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R}) / X_f(\varphi) = 0, \forall f \in C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})\}$.

Proof. Under the canonical identification defined by the scalar product between $\mathfrak{g}$ and $\mathfrak{g}^*$, the coadjoint action $\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ induces an action of the Lie group G on $\mathfrak{g}$, which will denote by $\widetilde{\mathrm{Ad}}^*$. Thus, we can define an action of G on the unit sphere $S^{n-1}(\mathfrak{g})$, $\overline{\mathrm{Ad}}^* : G \times S^{n-1}(\mathfrak{g}) \to S^{n-1}(\mathfrak{g})$, given by $\overline{\mathrm{Ad}}^*(g, \xi) = \widetilde{\mathrm{Ad}}^*(g, \xi)/\|\widetilde{\mathrm{Ad}}^*(g, \xi)\|$. This last action induces a representation of $\mathfrak{g}$ on the vector space $C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})$ defined by

$$(\xi, \varphi) \in \mathfrak{g} \times C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R}) \to \xi_{S^{n-1}(\mathfrak{g})}(\varphi) \in C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R}),$$

$\xi_{S^{n-1}(\mathfrak{g})}$ being the infinitesimal generator, with respect to the action $\overline{\mathrm{Ad}}^*$, associated to $\xi \in \mathfrak{g}$. Moreover, using a result in [20], we deduce that $\xi_{S^{n-1}(\mathfrak{g})}$ is the Hamiltonian vector field on $S^{n-1}(\mathfrak{g})$ associated to the function $\langle \xi, \cdot \rangle : S^{n-1}(\mathfrak{g}) \to \mathbb{R}$ given by $\langle \xi, \cdot \rangle(\eta) = \langle \xi, \eta \rangle$, that is,

$$\xi_{S^{n-1}(\mathfrak{g})} = X_{\langle \xi, \cdot \rangle}. \tag{4.24}$$

The above representation allows us to consider the differential complex $(C^*(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})), \partial)$ and its cohomology $H^*(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R}))$.

We will show that $H^k_{\mathrm{LJ}}(S^{n-1}(\mathfrak{g})) \cong H^k(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R}))$, for all k.

Let $C^k_{\mathrm{HCE}}(S^{n-1}(\mathfrak{g}))$ be the space of k-cochains in the H–Chevalley–Eilenberg complex of $S^{n-1}(\mathfrak{g})$. We define the homomorphism $\mu^k : C^k_{\mathrm{HCE}}(S^{n-1}(\mathfrak{g})) \to C^k(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R}))$ by

$$(\mu^k(c^k))(\xi_1, \dots, \xi_k) = c^k(\langle \xi_1, \cdot \rangle, \dots, \langle \xi_k, \cdot \rangle) \tag{4.25}$$

for all $c^k \in C^k_{\mathrm{HCE}}(S^{n-1}(\mathfrak{g}))$ and $\xi_1, \dots, \xi_k \in \mathfrak{g}$.

Now, consider the homomorphism of $C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})$ modules

$$\Phi^k : \mathcal{V}^k(S^{n-1}(\mathfrak{g})) \oplus \mathcal{V}^{k-1}(S^{n-1}(\mathfrak{g})) \to C^k(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R}))$$

defined by

$$\Phi^k = \mu^k \circ j^k, \tag{4.26}$$

$j^k : \mathcal{V}^k(S^{n-1}(\mathfrak{g})) \oplus \mathcal{V}^{k-1}(S^{n-1}(\mathfrak{g})) \to C^k_{\mathrm{HCE}}(S^{n-1}(\mathfrak{g}))$ being the mapping given by (3.1).

A direct computation shows that

$$\begin{aligned}(\Phi^k(P, Q))(\xi_1, \dots, \xi_k)(\xi) &= P(\mathrm{d}\langle \xi_1, \cdot\rangle, \dots, \mathrm{d}\langle \xi_k, \cdot\rangle)(\xi) \\ &\quad + \sum_{i=1}^{k}(-1)^{i+1}\langle \xi_i, \xi\rangle Q(\mathrm{d}\langle \xi_1, \cdot\rangle, \dots, \widehat{\mathrm{d}\langle \xi_i, \cdot\rangle}, \dots, \mathrm{d}\langle \xi_k, \cdot\rangle)(\xi) \\ &= (\widetilde{(P, Q)}(\mathrm{d}\tilde{\xi}_1, \dots, \mathrm{d}\tilde{\xi}_k))(\xi, 0)\end{aligned} \tag{4.27}$$

for all $(P, Q) \in \mathcal{V}^k(S^{n-1}(\mathfrak{g})) \oplus \mathcal{V}^{k-1}(S^{n-1}(\mathfrak{g}))$, $\xi_1, \dots, \xi_k \in \mathfrak{g}$ and $\xi \in S^{n-1}(\mathfrak{g})$, where $\tilde{\xi}_i$ $(i = 1, \dots, k)$ is the function on $S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ given by (4.23) and $\widetilde{(P, Q)}$ is the k-vector on $S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ defined by

$$\widetilde{(P, Q)} = \mathrm{e}^{-kt}\left(P + \frac{\partial}{\partial t} \wedge Q\right). \tag{4.28}$$

On the other hand, if $\{\cdot, \cdot\}$ is the Jacobi bracket on $S^{n-1}(\mathfrak{g})$, then using (2.3) and the expression of $\{\cdot, \cdot\}$ in terms of global coordinates on $\mathfrak{g}$ obtained from an orthonormal basis, one can prove that for all $\xi, \eta \in \mathfrak{g}$

$$\{\langle \xi, \cdot\rangle, \langle \eta, \cdot\rangle\} = \langle [\xi, \eta], \cdot\rangle.$$

This fact, (4.24) and (4.25) imply that the mappings μ^k induce a homomorphism between the complexes $(C^*_{\mathrm{HCE}}(S^{n-1}(\mathfrak{g})), \partial_H)$ and $(C^*(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})), \partial)$. Thus, the mappings Φ^k induce a homomorphism between $(\mathcal{V}^*(S^{n-1}(\mathfrak{g})) \oplus \mathcal{V}^{*-1}(S^{n-1}(\mathfrak{g})), \sigma)$ and $(C^*(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})), \partial)$ (see (4.26) and results of the Section 3).

To show that Φ^k is a monomorphism, we suppose that $\Phi^k(P, Q) = 0$. Then, from (4.28) and Lemma 4.11, it follows that, $\mathcal{L}_{\partial/\partial t}\widetilde{(P, Q)} = -k\widetilde{(P, Q)}$, and $(\partial/\partial t)(\widetilde{(P, Q)}(\mathrm{d}\tilde{\xi}_1, \dots, \mathrm{d}\tilde{\xi}_k)) = 0$, for all $\xi_1, \dots, \xi_k \in \mathfrak{g}$, where $\tilde{\xi}_i$ $(i = 1, \dots, k)$ is the real C^∞-function on $S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ given by (4.23). Therefore, using these facts and (4.27), we deduce that $0 = \widetilde{(P, Q)} = \mathrm{e}^{-kt}(P + (\partial/\partial t) \wedge Q)$. Thus, $P = 0$ and $Q = 0$.

Next, we will see that Φ^k is an epimorphism. Let $c^k : \mathfrak{g} \times \cdots^k \cdots \times \mathfrak{g} \to C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})$ be a $C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})$-valued k-cochain. We define a k-vector R on $S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ characterized by the condition

$$R(\mathrm{d}\tilde{\xi}_1, \dots, \mathrm{d}\tilde{\xi}_k)(\xi, t) = \mathrm{e}^{kt}(c^k(\xi_1, \dots, \xi_k)(\xi)) \tag{4.29}$$

for all $\xi_1, \dots, \xi_k \in \mathfrak{g}$ and $(\xi, t) \in S^{n-1}(\mathfrak{g}) \times \mathbb{R}$.

From Lemma 4.11 and (4.29), we deduce that R is well-defined and that $\mathcal{L}_{\partial/\partial t}R = 0$. This implies that

$$R = P + \frac{\partial}{\partial t} \wedge Q, \tag{4.30}$$

with $(P, Q) \in \mathcal{V}^k(S^{n-1}(\mathfrak{g})) \oplus \mathcal{V}^{k-1}(S^{n-1}(\mathfrak{g}))$. Moreover, from (4.27)–(4.30), it follows that $\Phi^k(P, Q) = c^k$. Thus, Φ^k is an epimorphism.

Therefore, we conclude that $H^k_{\mathrm{LJ}}(S^{n-1}(\mathfrak{g})) \cong H^k(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R}))$, for all k.

Now, if we apply a general result of Ginzburg and Weinstein (see Theorem 3.5 of [7]; see also [27]), we obtain that $H^k(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})) \cong H^k(\mathfrak{g}) \otimes \overline{\mathrm{Inv}}$, where $\overline{\mathrm{Inv}}$ is the algebra of G-invariant functions on $S^{n-1}(\mathfrak{g})$ with respect to the action $\overline{\mathrm{Ad}}^*$.

Finally, from (4.24) and since the characteristic foliation of $S^{n-1}(\mathfrak{g})$ is generated by the set of Hamiltonian vector fields $\{X_{\langle \xi, \cdot \rangle} / \xi \in \mathfrak{g}\}$, we deduce that $\overline{\mathrm{Inv}} = \mathrm{Inv}$. □

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